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Two-dimensional centrally extended quantum Galilei groups and their algebras

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Abstract. Deformations of the two-dimensional centrally extended Galilei group are constructed. The corresponding quantum Lie algebras are found.

1. Introduction

During the last few years, some authors have studied the problem of deformations of spacetime symmetry groups [1–4]. It is well known that the notion of a quantum group is closely related to that of Lie bialgebra and the Lie–Poisson group (see, for example, [5, 6]). In recent paper [7] we classified all non-equivalent Lie–Poisson structures for the centrally extended two-dimensional Galilei group. The central extension of Galilei algebra by the mass operator admits 26 inequivalent Lie bialgebra structures and the corresponding Lie–Poisson structures.

In the present paper we quantize the structures obtained in [7]. As a result we obtain Hopf algebras which provide quantum deformations of the centrally extended Galilei group. We also find (by duality relations; see also [5, 8]) the corresponding quantum Lie algebras.

2. Deformations of the two-dimensional centrally extended Galilei group

The classical centrally extended Galilei group is defined as a set of elements

$$g = (m, \tau, v, a) \tag{1}$$

where τ is time translation, a and v are space translation and Galilean boost, respectively, subject to the following multiplication law:

$$g'g = \left(m' + m - \tfrac{1}{2}v'^2\tau - av', \tau' + \tau, v' + v, a' + a + \tau v'\right). \tag{2}$$

The multiplication rule defines coproducts of m, v, τ and a:

$$\begin{aligned}
\triangle(m) &= m \otimes I + I \otimes m - \tfrac{1}{2}v^2 \otimes \tau - v \otimes a \\
\triangle(\tau) &= \tau \otimes I + I \otimes \tau \\
\triangle(a) &= a \otimes I + I \otimes a + v \otimes \tau \\
\triangle(v) &= v \otimes I + I \otimes v.
\end{aligned} \tag{3}$$

The antipode and counit can be also read off from equation (2):

$$\begin{aligned} S(m) &= -m + \tfrac{1}{2}v^2\tau - av \\ S(\tau) &= -\tau \\ S(a) &= -a + v\tau \\ S(v) &= -v \end{aligned} \qquad (4)$$

$$\varepsilon(m) = 0 \qquad \varepsilon(\tau) = 0 \qquad \varepsilon(a) = 0 \qquad \varepsilon(v) = 0. \qquad (5)$$

The Galilei group is a real Lie group. The reality condition can be described by introducing the following *-structure:

$$m^* = m \qquad \tau^* = \tau \qquad a^* = a \qquad v^* = v. \qquad (6)$$

The starting point to obtain the two-dimensional quantum Galilei group is the Lie–Poisson structures on it. In [7] we have found all relevant Poisson structures.

In order to obtain the corresponding quantum group we make a replacement

$$\{\, ,\,\} = \frac{1}{\mathrm{i}\kappa}[\, ,\,]. \qquad (7)$$

Where κ is an arbitrary parameter. The quantization procedure applied to the Lie–Poisson structures classified in [7] yields the non-commutative structures listed in table 1.

For all the above cases we have

$$[v, \tau] = 0. \qquad (8)$$

We have checked that the commutators listed above satisfy the Jacobi identities. We have also verified that equations (3)–(6), together with table 1 and (8) define *-Hopf algebras which provide the deformations of the two-dimensional centrally extended Galilei group.

3. The quantum Lie algebra and coalgebra

In this section we find all quantum Lie algebras corresponding to the quantum Galilei groups defined in section 2. To this end we use the Hopf algebra duality rules. On the classical level the generators of the Lie algebra of the Galilei group can be defined by the following global parametrization of the group element:

$$g = \mathrm{e}^{\mathrm{i}mM}\mathrm{e}^{-\mathrm{i}\tau H}\mathrm{e}^{\mathrm{i}aP}\mathrm{e}^{\mathrm{i}vK}. \qquad (9)$$

Here we adopt this definition as well as the classical duality relation

$$\langle \varphi, X\rangle = -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\varphi\big(\mathrm{e}^{\mathrm{i}tX}\big)|_{t=0}. \qquad (10)$$

The group algebra is generated by the set of elements of the form

$$\varphi^{\alpha\beta\gamma\sigma} = m^\alpha\tau^\beta a^\gamma v^\sigma \qquad (11)$$

where $\alpha, \beta, \gamma, \sigma \geqslant 1$. By applying the classical duality rules we obtain

$$\begin{aligned} \langle M, m^\alpha\tau^\beta a^\gamma v^\sigma\rangle &= -\mathrm{i}\delta_{1\alpha}\delta_{0\beta}\delta_{0\gamma}\delta_{0\sigma} \\ \langle H, m^\alpha\tau^\beta a^\gamma v^\sigma\rangle &= \mathrm{i}\delta_{0\alpha}\delta_{1\beta}\delta_{0\gamma}\delta_{0\sigma} \\ \langle P, m^\alpha\tau^\beta a^\gamma v^\sigma\rangle &= -\mathrm{i}\delta_{0\alpha}\delta_{0\beta}\delta_{1\gamma}\delta_{0\sigma} \\ \langle K, m^\alpha\tau^\beta a^\gamma v^\sigma\rangle &= -\mathrm{i}\delta_{0\alpha}\delta_{0\beta}\delta_{0\gamma}\delta_{1\sigma}. \end{aligned} \qquad (12)$$

Table 1.

	$[v, a]$	$[v, m]$	$[\tau, a]$	$[\tau, m]$	$[a, m]$	
1		$\mathrm{i}\frac{1}{\chi_1}v$			$\mathrm{i}\frac{1}{\chi_1}a + \mathrm{i}\frac{1}{\chi_2}v$	$\frac{1}{\chi_1} = \frac{{v_0}^2\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{{v_0}^2{\tau_0}^2}{\kappa}$
2		$\mathrm{i}\frac{1}{\chi_1}v$	$\mathrm{i}\frac{1}{\chi_2}v$	$-\frac{1}{2}\mathrm{i}\frac{1}{\chi_2}v^2$	$\mathrm{i}\frac{1}{\chi_1}a - \frac{1}{6}\mathrm{i}\frac{1}{\chi_2}v^3$	$\frac{1}{\chi_1} = \frac{{v_0}^2\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{{\tau_0}^2}{\kappa}$
3				$\mathrm{i}\frac{1}{\chi_1}v$	$\frac{1}{2}\mathrm{i}\frac{1}{\chi_1}v^2 - \mathrm{i}\frac{1}{\chi_2}\tau$	$\frac{1}{\chi_1} = \frac{v_0{\tau_0}^2}{\kappa}$, $\frac{1}{\chi_2} = \frac{{v_0}^3\tau_0}{\kappa}$
4	$\mathrm{i}\frac{1}{\chi_1}v$	$-\frac{1}{2}\mathrm{i}\frac{1}{\chi_1}v^2$		$\mathrm{i}\frac{1}{\chi_2}v$	$\mathrm{i}\frac{1}{\chi_3}v - \mathrm{i}\frac{1}{\chi_1}m + \frac{1}{2}\mathrm{i}\frac{1}{\chi_2}v^2$	$\frac{1}{\chi_1} = \frac{v_0\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{v_0{\tau_0}^2}{\kappa}$ $\frac{1}{\chi_3} = \frac{\varepsilon{v_0}^2{\tau_0}^2}{\kappa}$
5			$\mathrm{i}\frac{1}{\chi_1}v$	$-\frac{1}{2}\mathrm{i}\frac{1}{\chi_1}v^2$	$-\mathrm{i}\frac{1}{\chi_2}\tau - \frac{1}{6}\mathrm{i}\frac{1}{\chi_1}v^3$	$\frac{1}{\chi_1} = \frac{{\tau_0}^2}{\kappa}$, $\frac{1}{\chi_2} = \frac{{v_0}^3\tau_0}{\kappa}$
6				$-\mathrm{i}\frac{1}{\chi_1}\tau$	$-\mathrm{i}\frac{1}{\chi_2}\tau - \mathrm{i}\frac{1}{\chi_1}a$	$\frac{1}{\chi_1} = \frac{{v_0}^2\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{{v_0}^3\tau_0}{\kappa}$
7		$\mathrm{i}\frac{1}{\chi_1}v$		$-\mathrm{i}\frac{1}{\chi_2}\tau$	$\mathrm{i}\left(\frac{1}{\chi_1} - \frac{1}{\chi_2}\right)a$	$\frac{1}{\chi_1} = \frac{{v_0}^2\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{\varepsilon{v_0}^2\tau_0}{\kappa}$
8		$\mathrm{i}\frac{1}{\chi_1}v$		$\mathrm{i}\frac{1}{\chi_1}\tau + \mathrm{i}\frac{1}{\chi_2}v$	$2\mathrm{i}\frac{1}{\chi_1}a + \frac{1}{2}\mathrm{i}\frac{1}{\chi_2}v^2$	$\frac{1}{\chi_1} = \frac{{v_0}^2\tau_0}{\kappa}$, $\frac{1}{\chi_2} = \frac{v_0{\tau_0}^2}{\kappa}$
9		$-\mathrm{i}\frac{1}{\chi_1}\tau$		$-\mathrm{i}\frac{1}{\chi_2}\tau$	$-\mathrm{i}\frac{1}{\chi_2}a - \frac{1}{2}\mathrm{i}\frac{1}{\chi_1}\tau^2 + \mathrm{i}\frac{1}{\chi_3}v$	$\frac{1}{\chi_1} = \frac{{v_0}^3}{\kappa}$, $\frac{1}{\chi_3} = \frac{{v_0}^2{\tau_0}^2}{\kappa}$ $\frac{1}{\chi_2} = \frac{\varepsilon{v_0}^2\tau_0}{\kappa}$
10		$-\mathrm{i}\frac{1}{\chi_1}\tau$		$-\mathrm{i}\frac{1}{\chi_2}\tau + \mathrm{i}\frac{1}{\chi_3}v$	$-\mathrm{i}\frac{1}{\chi_2}a - \frac{1}{2}\mathrm{i}\frac{1}{\chi_1}\tau^2 + \frac{1}{2}\mathrm{i}\frac{1}{\chi_3}v^2$	$\frac{1}{\chi_1} = \frac{{v_0}^3}{\kappa}$, $\frac{1}{\chi_3} = \frac{v_0{\tau_0}^2}{\kappa}$ $\frac{1}{\chi_2} = \frac{\varepsilon{v_0}^2\tau_0}{\kappa}$
11			$-\mathrm{i}\frac{1}{\chi}\tau$	$\mathrm{i}\frac{1}{\chi}a$	$-\mathrm{i}\frac{1}{\chi}m$	$\frac{1}{\chi} = \frac{v_0\tau_0}{\kappa}$

In order to define dual algebra structures we use the duality relations

$$\begin{aligned} \langle \varphi, XY \rangle &= \langle \triangle\varphi, X \otimes Y \rangle \\ \langle \varphi\psi, X \rangle &= \langle \varphi \otimes \psi, \triangle(X) \rangle. \end{aligned} \tag{13}$$

The *-structure on the dual Hopf algebra can be defined by the formula

$$\langle X^*, \varphi \rangle = \langle X, S^{-1}\left(\varphi^*\right) \rangle \tag{14}$$

provided the following relation holds:

$$S^{-1}(X) = [S\left(X^*\right)]^*. \tag{15}$$

In order to take care of all elements of the form (11), let us introduce a generating function depending on four real parameters $\mu, \nu, \varrho, \kappa$ (see, however, the appendix):

$$\varphi(\mu, \nu, \rho, \kappa) = \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\rho a}\mathrm{e}^{\kappa v}. \tag{16}$$

Then we have

$$\begin{aligned}
&\langle M, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\rho a}\mathrm{e}^{\kappa v}\rangle = -\mathrm{i}\mu \\
&\langle H, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\rho a}\mathrm{e}^{\kappa v}\rangle = \mathrm{i}\nu \\
&\langle P, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\rho a}\mathrm{e}^{\kappa v}\rangle = -\mathrm{i}\varrho \\
&\langle K, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\rho a}\mathrm{e}^{\kappa v}\rangle = -\mathrm{i}\kappa.
\end{aligned} \tag{17}$$

The duality rules (13) and long and tedious calculations lead us to the following structure of quantum Lie algebras:

1.

$$\begin{aligned}
&[K, H] = \mathrm{i}P \\
&[K, P] = \tfrac{1}{2}\mathrm{i}\chi_1\left(1 - \mathrm{e}^{-2M/\chi_1}\right) \\
&\Delta M = I \otimes M + M \otimes I \\
&\Delta H = I \otimes H + H \otimes I \\
&\Delta P = I \otimes P + P \otimes \mathrm{e}^{-M/\chi_1} \\
&\Delta K = I \otimes K + K \otimes \mathrm{e}^{-M/\chi_1} - \frac{1}{\chi_2} P \otimes M\mathrm{e}^{-M/\chi_1} \\
&S(M) = -M \\
&S(H) = -H \\
&S(P) = -P\mathrm{e}^{M/\chi_1} \\
&S(K) = -K\mathrm{e}^{M/\chi_1} - \frac{1}{\chi_2} PM\mathrm{e}^{M/\chi_1}.
\end{aligned} \tag{18}$$

2.

$$\begin{aligned}
&[K, H] = \mathrm{i}P \\
&[K, P] = \tfrac{1}{2}\mathrm{i}\chi_1\left(1 - \mathrm{e}^{-2M/\chi_1}\right) \\
&\Delta M = I \otimes M + M \otimes I \\
&\Delta H = I \otimes H + H \otimes I \\
&\Delta P = I \otimes P + P \otimes \mathrm{e}^{-M/\chi_1} \\
&\Delta K = I \otimes K + K \otimes \mathrm{e}^{-M/\chi_1} - \frac{1}{\chi_2} P \otimes H\mathrm{e}^{-M/\chi_1} \\
&S(M) = -M \\
&S(H) = -H \\
&S(P) = -P\mathrm{e}^{M/\chi_1} \\
&S(K) = -K\mathrm{e}^{M/\chi_1} - \frac{1}{\chi_2} PH\mathrm{e}^{M/\chi_1}.
\end{aligned} \tag{19}$$

3.

$$\begin{aligned}
&[K, P] = \mathrm{i}M \\
&[K, H] = \mathrm{i}P - \tfrac{1}{2}\mathrm{i}\frac{1}{\chi_2}M^2 \\
&\Delta M = I \otimes M + M \otimes I \\
&\Delta H = I \otimes H + H \otimes I - \frac{1}{\chi_2}P \otimes M \\
&\Delta P = I \otimes P + P \otimes I \\
&\Delta K = I \otimes K + K \otimes I + \frac{1}{\chi_1}H \otimes M - \tfrac{1}{2}\frac{1}{\chi_1}\frac{1}{\chi_2}P \otimes M^2 \\
&S(M) = -M \\
&S(H) = -H - \frac{1}{\chi_2}PM \\
&S(P) = -P \\
&S(K) = -K + \frac{1}{\chi_1}HM + \frac{1}{2}\frac{1}{\chi_1}\frac{1}{\chi_2}PM^2.
\end{aligned} \tag{20}$$

4.

$$\begin{aligned}
&[K, M] = \tfrac{1}{2}\mathrm{i}\frac{1}{\chi_1}M^2 \\
&[K, P] = \mathrm{i}M \\
&[K, H] = -\mathrm{i}\chi_1\left(1 - \mathrm{e}^{P/\chi_1}\right) \\
&\Delta M = \mathrm{e}^{P/\chi_1} \otimes M + M \otimes I \\
&\Delta H = I \otimes H + H \otimes I \\
&\Delta P = I \otimes P + P \otimes I \\
&\Delta K = \mathrm{e}^{P/\chi_1} \otimes K + K \otimes I + \frac{1}{\chi_2}H\mathrm{e}^{P/\chi_1} \otimes M - \frac{1}{\chi_3}P\mathrm{e}^{P/\chi_1} \otimes M \\
&S(M) = -M\mathrm{e}^{-P/\chi_1} \\
&S(H) = -H \\
&S(P) = -P \\
&S(K) = -K\mathrm{e}^{-P/\chi_1} + \frac{1}{\chi_2}HM\mathrm{e}^{-P/\chi_1} - \frac{1}{\chi_3}PM\mathrm{e}^{-P/\chi_1}.
\end{aligned} \tag{21}$$

5.

$$\begin{aligned}
&[K, P] = \mathrm{i}M \\
&[K, H] = \mathrm{i}P - \tfrac{1}{2}\mathrm{i}\frac{1}{\chi_2}M^2 \\
&\Delta M = I \otimes M + M \otimes I \\
&\Delta H = I \otimes H + H \otimes I - \frac{1}{\chi_2}P \otimes M
\end{aligned}$$

$$\Delta P = I \otimes P + P \otimes I$$
$$\Delta K = I \otimes K + K \otimes I + \frac{1}{2}\frac{1}{\chi_1}\frac{1}{\chi_2}P^2 \otimes M - \frac{1}{\chi_1}P \otimes H \tag{22}$$
$$S(M) = -M$$
$$S(H) = -H - \frac{1}{\chi_2}PM$$
$$S(P) = -P$$
$$S(K) = -K - \frac{1}{\chi_1}PM - \frac{1}{2}\frac{1}{\chi_1}\frac{1}{\chi_2}MP^2.$$

6.

$$[K, H] = \mathrm{i}P - \mathrm{i}\frac{\chi_1}{\chi_2}M\mathrm{e}^{M/\chi_1} + \mathrm{i}\frac{\chi_1^2}{\chi_2}\left(\mathrm{e}^{M/\chi_1} - 1\right)$$
$$[K, P] = -\mathrm{i}\chi_1\left(\mathrm{e}^{M/\chi_1} - 1\right)$$
$$\Delta M = I \otimes M + M \otimes I$$
$$\Delta H = I \otimes H + H \otimes \mathrm{e}^{M/\chi_1} - \frac{1}{\chi_2}P \otimes M\mathrm{e}^{M/\chi_1}$$
$$\Delta P = I \otimes P + P \otimes \mathrm{e}^{M/\chi_1} \tag{23}$$
$$\Delta K = I \otimes K + K \otimes I$$
$$S(M) = -M$$
$$S(H) = -H\mathrm{e}^{-M/\chi_1} - \frac{1}{\chi_2}PM\mathrm{e}^{-M/\chi_1}$$
$$S(P) = -P\mathrm{e}^{-M/\chi_1}$$
$$S(K) = -K.$$

7.

$$[K, H] = \mathrm{i}P$$
$$[K, P] = -\frac{\mathrm{i}}{(2/\chi_1 - 1/\chi_2)}\left(1 - \mathrm{e}^{-(2/\chi_1 - 1/\chi_2)M}\right)$$
$$\Delta M = I \otimes M + M \otimes I$$
$$\Delta H = I \otimes H + H \otimes \mathrm{e}^{M/\chi_2}$$
$$\Delta P = I \otimes P + P \otimes \mathrm{e}^{(1/\chi_2 - 1/\chi_1)M} \tag{24}$$
$$\Delta K = I \otimes K + K \otimes \mathrm{e}^{-M/\chi_1}$$
$$S(M) = -M$$
$$S(H) = -H\mathrm{e}^{-M/\chi_2}$$
$$S(P) = -P\mathrm{e}^{(1/\chi_1 - 1/\chi_2)M}$$
$$S(K) = -K\mathrm{e}^{M/\chi_1}.$$

8.

$$[K, H] = \mathrm{i}P$$

$$
\begin{aligned}
&[K, P] = \tfrac{1}{3}\mathrm{i}\chi_1\big(1 - \mathrm{e}^{-3M/\chi_1}\big)\\
&\Delta M = I \otimes M + M \otimes I\\
&\Delta H = I \otimes H + H \otimes \mathrm{e}^{-M/\chi_1}\\
&\Delta P = I \otimes P + P \otimes \mathrm{e}^{-2M/\chi_1}\\
&\Delta K = I \otimes K + K \otimes \mathrm{e}^{-M/\chi_1} + \frac{1}{\chi_2} H \otimes M\mathrm{e}^{-M/\chi_1} \qquad (25)\\
&S(M) = -M\\
&S(H) = -H\mathrm{e}^{M/\chi_1}\\
&S(P) = -P\mathrm{e}^{2M/\chi_1}\\
&S(K) = -K\mathrm{e}^{M/\chi_1} + \frac{1}{\chi_2} HM\mathrm{e}^{M/\chi_1}.
\end{aligned}
$$

9.

$$
\begin{aligned}
&[K, H] = \mathrm{i}P + \mathrm{i}\frac{\chi_2^2}{\chi_1\chi_3} M\mathrm{e}^{M/\chi_2} - \frac{1}{2}\mathrm{i}\frac{\chi_2^3}{\chi_1\chi_3}\left(\mathrm{e}^{2M/\chi_2} - 1\right)\\
&[K, P] = \mathrm{i}\chi_2\left(\mathrm{e}^{M/\chi_2} - 1\right)\\
&[H, P] = -\frac{1}{2}\mathrm{i}\frac{\chi_2^2}{\chi_1}\left(\mathrm{e}^{2M/\chi_2} - 1\right) + \mathrm{i}\frac{\chi_2^2}{\chi_1}\left(\mathrm{e}^{M/\chi_2} - 1\right)
\end{aligned}
$$

$$
\begin{aligned}
&\Delta M = I \otimes M + M \otimes I\\
&\Delta H = I \otimes H + H \otimes \mathrm{e}^{M/\chi_2} - \frac{\chi_2}{\chi_1} K \otimes \mathrm{e}^{M/\chi_2} + \frac{\chi_2}{\chi_1} K \otimes I\\
&\qquad + \frac{\chi_2}{\chi_1\chi_3} P \otimes M\mathrm{e}^{M/\chi_2} + \frac{\chi_2^2}{\chi_1\chi_3} P \otimes I - \frac{\chi_2^2}{\chi_1\chi_3} P \otimes \mathrm{e}^{M/\chi_2} \qquad (26)\\
&\Delta P = I \otimes P + P \otimes \mathrm{e}^{M/\chi_2}\\
&\Delta K = I \otimes K + K \otimes I + \frac{\chi_2}{\chi_3} P \otimes I - \frac{\chi_2}{\chi_3} P \otimes \mathrm{e}^{M/\chi_2}
\end{aligned}
$$

$$
\begin{aligned}
&S(M) = -M\\
&S(H) = -H\mathrm{e}^{-M/\chi_2} - \frac{\chi_2}{\chi_1} K + \frac{\chi_2}{\chi_1} K\mathrm{e}^{-M/\chi_2} + \frac{\chi_2^2}{\chi_1\chi_3} P\mathrm{e}^{-M/\chi_2} - \frac{\chi_2^2}{\chi_1\chi_3} P + \frac{\chi_2}{\chi_1\chi_3} PM\mathrm{e}^{-M/\chi_2}\\
&S(P) = -P\mathrm{e}^{-M/\chi_2}\\
&S(K) = -K + \frac{\chi_2}{\chi_3} P\mathrm{e}^{-M/\chi_2} - \frac{\chi_2}{\chi_3} P.
\end{aligned}
$$

10.

$$[K, H] = \mathrm{i}P$$

$$[K, P] = \frac{\mathrm{i}\left(-\frac{1}{2}(1/\chi_2) + \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}{\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\left(\frac{3}{2}(1/\chi_2) + \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}$$

$$\times\left(\mathrm{e}^{\left(\frac{3}{2}(1/\chi_2)+\frac{1}{2}\sqrt{1/\chi_2^2-4(1/\chi_1)/\chi_3}\right)M} - 1\right)$$

$$+\frac{\mathrm{i}\left(\frac{1}{2}(1/\chi_2) + \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}{\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\left(\frac{3}{2}(1/\chi_2) - \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}$$

$$\times\left(\mathrm{e}^{\left(\frac{3}{2}(1/\chi_2)-\frac{1}{2}\sqrt{1/\chi_2^2-4(1/\chi_1)/\chi_3}\right)M} - 1\right)$$

$$[H, P] = \frac{\mathrm{i}}{\chi_1\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\left(\frac{3}{2}(1/\chi_2) - \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}$$

$$\times\left(\mathrm{e}^{\left(\frac{3}{2}(1/\chi_2)-\frac{1}{2}\sqrt{1/\chi_2^2-4(1/\chi_1)/\chi_3}\right)M} - 1\right)$$

$$-\frac{\mathrm{i}}{\chi_1\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\left(\frac{3}{2}(1/\chi_2) + \frac{1}{2}\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}\right)}$$

$$\times\left(\mathrm{e}^{\left(\frac{3}{2}(1/\chi_2)+\frac{1}{2}\sqrt{1/\chi_2^2-4(1/\chi_1)/\chi_3}\right)M} - 1\right)$$

$$\Delta M = I \otimes M + M \otimes I$$

$$\Delta H = I \otimes H + H \otimes \mathrm{e}^{\frac{1}{2}M/\chi_2}\left[\cosh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}}M\right)\right.$$

$$\left. - \frac{1}{\chi_2\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}\sinh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}}M\right)\right]$$

$$+2\frac{1}{\chi_1\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}K \otimes \mathrm{e}^{\frac{1}{2}M/\chi_2}\sinh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}}M\right)$$

$$\Delta P = I \otimes P + P \otimes \mathrm{e}^{M/\chi_2}$$

$$\Delta K = I \otimes K + K \otimes \mathrm{e}^{\frac{1}{2}M/\chi_2}\left[\cosh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right. \tag{27}$$

$$\left. + \frac{1}{\chi_2\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}\sinh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right]$$

$$-2\frac{1}{\chi_3\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}H \otimes \mathrm{e}^{\frac{1}{2}M/\chi_2}\sinh\left(-\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)$$

$$S(M) = -M$$

$$S(H) = -H\mathrm{e}^{-\frac{1}{2}M/\chi_2}\left[\cosh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right.$$

$$\left. - \frac{1}{\chi_2\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right]$$

$$-2\frac{1}{\chi_1\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}K\mathrm{e}^{-\frac{1}{2}M/\chi_2}\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)$$

$$S(P) = -P\mathrm{e}^{-M/\chi_2}$$

$$S(K) = -K\mathrm{e}^{-\frac{1}{2}M/\chi_2}\left[\cosh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right.$$

$$\left. + \frac{1}{\chi_2\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right)\right]$$

$$+2\frac{1}{\chi_3\sqrt{1/\chi_2^2 - 4(1/\chi_1)/\chi_3}}H\mathrm{e}^{-\frac{1}{2}M/\chi_2}\sinh\left(\frac{1}{2}\sqrt{\frac{1}{\chi_2^2} - 4\frac{1}{\chi_1}\frac{1}{\chi_3}M}\right).$$

11.

$$[K, M] = \frac{1}{2}\mathrm{i}\frac{1}{\chi}M^2$$

$$[K, P] = \mathrm{i}M$$

$$[K, H] = \mathrm{i}\chi\left(1 - \mathrm{e}^{-P/\chi}\right) - \mathrm{i}\frac{1}{\chi}MH$$

$$\Delta M = M \otimes I + \frac{\mathrm{e}^{P/\chi} \otimes M}{1 - \frac{1}{2}(1/\chi^2)H\mathrm{e}^{P/\chi} \otimes M}$$

$$\Delta H = \mathrm{e}^{-P/\chi} \otimes H + H \otimes I - \frac{1}{\chi^2}H \otimes MH + \frac{1}{4}\frac{1}{\chi^4}H^2\mathrm{e}^{P/\chi} \otimes M^2H$$

$$-\frac{1}{2}\frac{1}{\chi^2}H^2\mathrm{e}^{P/\chi} \otimes M \tag{28}$$

$$\Delta P = I \otimes P + P \otimes I - 2\chi \ln\left(1 - \frac{1}{2}\frac{1}{\chi^2} H \mathrm{e}^{P/\chi} \otimes M\right)$$

$$\Delta K = I \otimes K + K \otimes I$$

$$S(M) = \frac{-M\mathrm{e}^{-P/\chi}}{1 - \frac{1}{2}(1/\chi^2)MH}$$

$$S(H) = -H\mathrm{e}^{P/\chi} + \frac{1}{2}\frac{1}{\chi^2} MH^2 \mathrm{e}^{P/\chi}$$

$$S(P) = -P + 2\chi \ln\left(\frac{1}{1 - \frac{1}{2}(1/\chi^2)HM}\right)$$

$$S(K) = -K.$$

The counits for all cases take the form

$$\varepsilon(M) = 0 \qquad \varepsilon(H) = 0 \qquad \varepsilon(P) = 0 \qquad \varepsilon(K) = 0. \tag{29}$$

4. Conclusions

We obtained a number of, in general multiparameter, deformations of the two-dimensional centrally extended Galilei group. They are described in table 1 together with equations (3)–(6) and (8). The corresponding quantum Lie algebras have also been found. They are listed in equations (18)–(29).

Recently, a paper has appeared by Ballesteros *et al* [9]. These authors have also studied the deformations of the Galilei algebra. All cases which have been obtained by Ballesteros *et al* are contained in our list. In two cases this equivalence becomes explicit only after some redefinition of generators.

The quantum deformations of Galilei group/algebra obtained above can serve to study dynamical models (classical as well as quantum ones) on non-commutative spacetime. It seems to us that they provide a better starting point to understanding the role of the non-commutativity of spacetime symmetries than their relativistic counterparts. This is because we can get rid of some additional difficulties related to quantum relativity (such as, for example, the impossibility of constructing non-trivial quantum dynamics with a fixed number of particles) and concentrate on the role of non-commutativity. These problems will be addressed in subsequent publications.

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Appendix

In this appendix, in order to illustrate the procedures used, we sketch the proof of equations (21).

This case is somehow special because it is more convenient to choose the generating function as follows (instead of equation (16)):

$$\varphi = \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}.$$

In order to determine coalgebra sector we use the duality relation

$$\langle \varphi\psi, X\rangle = \langle \varphi \otimes \psi, \triangle(X)\rangle.$$

To this end, we calculate

$$\begin{aligned}
\mathrm{e}^{\mu' m}\mathrm{e}^{\nu'\tau}\mathrm{e}^{\kappa' v}\mathrm{e}^{\rho' a}\mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a} &= \mathrm{e}^{(\mu'+\mu \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1})m}\mathrm{e}^{\left(\nu'+\nu\right)\tau}\\
&\quad\times \exp\left(\kappa \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1} + \kappa' \frac{1}{\frac{1}{2}\mathrm{i}(1/\chi_1)\mu \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1} v + 1}\right) v\\
&\quad+ \left(2(\chi_1/\chi_2)\nu' + 2(\chi_1/\chi_3)\rho'\right)\ln\left(\tfrac{1}{2}\mathrm{i}(1/\chi_1) v\mu \mathrm{e}^{-\mathrm{i}\rho'/\chi_1} + 1\right)\\
&\quad\times \mathrm{e}^{\mathrm{O}\left(v^2,\ldots\right)}\mathrm{e}^{\left(\rho'+\rho\right)a}
\end{aligned}$$

where $\mathrm{O}\left(v^2,\ldots\right)$ means a function depending on v^2 and higher powers of v.

Taking the first power of v in exponentials we obtain the following result:

$$\begin{aligned}
\varphi\psi &= \mathrm{e}^{(\mu'+\mu \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1})m}\mathrm{e}^{(\nu'+\nu)\tau}\\
&\quad\times \mathrm{e}^{(\kappa \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1}+\kappa'+\mathrm{i}(1/\chi_2)\mu\nu' \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1}+\mathrm{i}(1/\chi_3)\mu\rho' \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1})v}\mathrm{e}^{\mathrm{O}(v^2,\ldots)}\mathrm{e}^{(\rho'+\rho)a}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\langle \Delta M, \mathrm{e}^{\mu' m}\mathrm{e}^{\nu'\tau}\mathrm{e}^{\kappa' v}\mathrm{e}^{\rho' a} \otimes \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\rangle &= \langle M, \mathrm{e}^{\mu' m}\mathrm{e}^{\nu'\tau}\mathrm{e}^{\kappa' v}\mathrm{e}^{\rho' a}\mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\rangle\\
&= -\mathrm{i}\left(\mu' + \mu \mathrm{e}^{-\mathrm{i}\varrho'/\chi_1}\right).
\end{aligned}$$

Using this and the corresponding formulae for H, P and K we arrive at the coalgebra described by equation (21).

Now using the duality relation $\langle\varphi, XY\rangle = \langle\triangle\varphi, X\otimes Y\rangle$ we can determine the algebra sector. We have

$$\Delta\varphi = \mathrm{e}^{\mu\left(m'+m-\frac{1}{2}v'^2\tau - av'\right)}\mathrm{e}^{\nu\left(\tau'+\tau\right)}\mathrm{e}^{\kappa\left(v'+v\right)}\mathrm{e}^{\rho\left(a'+a+\tau v'\right)}.$$

In order to deal with the above expression we first decompose

$$\mathrm{e}^{\mu\left(m'+m-\frac{1}{2}v'^2\tau - av'\right)}.$$

To this end let

$$f(\mu) = \mathrm{e}^{-\mu m'}\mathrm{e}^{\mu\left(m'+m-\frac{1}{2}v'^2\tau - av'\right)}.$$

Then

$$\Delta\varphi = \mathrm{e}^{\mu m'} f \mathrm{e}^{\nu\left(\tau'+\tau\right)}\mathrm{e}^{\kappa\left(v'+v\right)}\mathrm{e}^{\rho\left(a'+a+\tau v'\right)}.$$

The function $f(\mu)$ obeys $f(0) = 1$, and

$$\dot{f} = \frac{\mathrm{d}f}{\mathrm{d}\mu} = \left(m - \frac{1}{2}\frac{\tau v'^2}{\left(1+\frac{1}{2}\mathrm{i}(1/\chi_1)\mu v'\right)^2} - \frac{av'}{1+\frac{1}{2}\mathrm{i}(1/\chi_1)\mu v'}\right) f.$$

The above equation cannot be solved in a standard way due to the fact that the terms appearing on the right-hand side do not commute. Therefore, we pass to the 'interaction picture' by letting

$$f = \mathrm{e}^{mX(v',\mu)}h.$$

Then

$$\dot{f} = m\dot{X}\mathrm{e}^{mX}h + \mathrm{e}^{mX}\dot{h}$$

and we select X in such a way that the terms containing m cancel. It can be checked that X should be of the form

$$X(v', \mu) = \mu\left(1 + \frac{1}{2}\mathrm{i}\frac{1}{\chi_1}\mu v'\right).$$

We now have

$$\mathrm{e}^{\mu\left(m'+m-\frac{1}{2}v'^2\tau-av'\right)} = \mathrm{e}^{\mu m'}\mathrm{e}^{\mu m\left(1+\frac{1}{2}\mathrm{i}\mu v'/\chi_1\right)}h.$$

Our next aim is to calculate h. We do this in the same way by writing out a relevant differential equation

$$\dot{h} = -\left(\frac{av'}{1+\frac{1}{2}\mathrm{i}(1/\chi_1)\mu v'} + \gamma\left(\tau, v, v', \mu\right)\right)h$$

and substituting

$$h = \mathrm{e}^{\lambda(v',\mu)a}$$

in order to deal with non-commutativity of the terms on the right-hand side.

Step by step we arrive at the following result:

$$\mathrm{e}^{\mu\left(m'+m-\frac{1}{2}v'^2\tau-av'\right)} = \mathrm{e}^{\mu m'}\mathrm{e}^{\mu m\left(1+\frac{1}{2}\mathrm{i}\mu v'/\chi_1\right)}\mathrm{e}^{2i\chi_1\ln\left(1+\frac{1}{2}\mathrm{i}\mu v'/\chi_1\right)a}$$
$$\times\exp\left[-\int_0^{\mu}\gamma\left(\frac{v}{\left(1+\frac{1}{2}\mathrm{i}\mu v'/\chi_1\right)^2}, v', \tau, \mu\right)\mathrm{d}\mu\right]$$

where the function γ is of third degree in v, v'.

Finally, $\Delta\varphi$ can be rewritten as

$$\Delta\varphi = \mathrm{e}^{\mu m'}\mathrm{e}^{\nu\tau'}\mathrm{e}^{\mu m}\mathrm{e}^{\frac{1}{2}\mathrm{i}(1/\chi_1)\mu^2v'm}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v'}\mathrm{e}^{\mathrm{O}(v'^2,v,\tau)}\mathrm{e}^{\kappa v\mathrm{e}^{\mathrm{i}\mu v'/\chi_1}}\mathrm{e}^{-\mu v'a}\mathrm{e}^{\varrho a}\mathrm{e}^{-\mathrm{i}\chi_1\left(1-\mathrm{e}^{-\mathrm{i}\varrho/\chi_1}\right)v'\tau}\mathrm{e}^{\varrho a'}.$$

Using the above formula, we may calculate

$$\langle KM, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\rangle = \langle K\otimes M, \Delta\left(\mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\right)\rangle = (-\mathrm{i})\,(-\mathrm{i})\left(\frac{1}{2}\mathrm{i}\frac{1}{\chi_1}\mu^2\right)$$

$$\langle MK, \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\rangle = \langle M\otimes K, \Delta\left(\mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\right)\rangle = 0.$$

Therefore,

$$\langle [K, M], \mathrm{e}^{\mu m}\mathrm{e}^{\nu\tau}\mathrm{e}^{\kappa v}\mathrm{e}^{\rho a}\rangle = -\frac{1}{2}\mathrm{i}\frac{1}{\chi_1}\mu^2.$$

Other commutators are calculated in a similar way.

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